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Index branch-and-bound algorithm for Lipschitz univariate global optimization with multiextremal constraints

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Abstract. In this paper, Lipschitz univariate constrained global optimization problems where both the objective function and constraints can be multiextremal are considered. The constrained problem is reduced to a discontinuous unconstrained problem by the index scheme without introducing additional parameters or variables. A Branch-and-Bound method that does not use derivatives for solving the reduced problem is proposed. The method either determines the infeasibility of the original problem or finds lower and upper bounds for the global solution. Not all the constraints are evaluated during every iteration of the algorithm, providing a significant acceleration of the search. Convergence conditions of the new method are established. Extensive numerical experiments are presented.

Key words: Global optimization, multiextremal constraints, branch-and-bound algorithms, index scheme.

1. Introduction

Global optimization problems arise in many real-life applications and were intensively studied during last decades (see, for example, Archetti and Schoen, 1984; Bomze et al., 1997; Breiman and Cutler, 1993; Evtushenko, 1992; Floudas and Pardalos, 1996; Horst and Pardalos, 1995; Horst and Tuy, 1996; Locatelli and Schoen, 1999; Lucidi, 1994; Mladineo, 1992; Pardalos and Rosen, 1990; Pintér, 1996; Strongin, 1978; Sun and Li, 1999; Törn and Žilinskas, 1989; Zhigljavsky, 1991, etc.). Particularly, univariate problems attract attention of many authors (see Calvin and Žilinskas, 1999; Hansen and Jaumard, 1995; Lamar, 1999; Locatelli and Schoen, 1995; MacLagan, Sturge, and Baritompa, 1996; Pijavskii, 1972; Sergeyev, 1998; Strongin, 1978; Wang and Chang, 1996) at least for two reasons. First, there exist a large number of applications where it is necessary to solve such problems (see Brooks, 1958; Hansen and Jaumard, 1995; Patwardhan, 1987; Ralston, 1985; Sergeyev et al., 1999; Strongin, 1978). Second, there exist numerous schemes (see, for example, Floudas and Pardalos, 1996; Horst and Pardalos, 1995; Horst and Tuy, 1996; Mladineo, 1992; Pardalos and Rosen, 1990; Pintér, 1996; Strongin, 1978) enabling to generalize to the multidimensional case the mathematical approaches developed to solve univariate problems.

In this paper we consider the global optimization problem

$$\min\{f(x) : x \in [a,b], \quad g_j(x) \leqslant 0, \quad 1 \leqslant j \leqslant m\}, \tag{1}$$

where $f(x)$ and $g_j(x), 1 \leqslant j \leqslant m$, are multiextremal Lipschitz functions (to unify the description process we shall use the designation $g_{m+1}(x) \triangleq f(x)$). Hereinafter we use the terminology 'multiextremal constraint' to highlight the fact that the constraints are described by multiextremal functions $g_j(x), 1 \leqslant j \leqslant m$, in the form (1) (of course, the same subregions of the interval $[a,b]$ may be defined in another way). In many practical problems the order of the constraints is fixed and not all the constraints are defined over the whole search region $[a,b]$ (if the order of the constraints is not a priori given, the user fixes his/her own ordering in a way). In the general case, a constraint $g_{j+1}(x)$ is defined only at subregions where $g_j(x) \leqslant 0$. We designate subdomains of the interval $[a,b]$ corresponding to the set of constraints from (1) as

$$Q_1 = [a,b], \quad Q_{j+1} = \{x \in Q_j : g_j(x) \leqslant 0\}, \quad 1 \leqslant j \leqslant m, \tag{2}$$

$$Q_1 \supseteq Q_2 \supseteq \ldots \supseteq Q_m \supseteq Q_{m+1}.$$

We introduce the number M such that

$$Q_M \neq \emptyset, \quad Q_{M+1} = Q_{M+2} \ldots = Q_{m+1} = \emptyset. \tag{3}$$

If the feasible region of the problem (1) is not empty then $Q_{m+1} \neq \emptyset$ and $M = m + 1$. In the opposite case M indicates the last subset Q_j from (2) such that $Q_j \neq \emptyset$.

We suppose in this paper that the functions $g_j(x), 1 \leqslant j \leqslant m + 1$, satisfy the Lipschitz condition in the form

$$\mid g_j(x') - g_j(x'') \mid \leqslant L_j \mid x' - x'' \mid, \quad x', x'' \in Q_j, \quad 1 \leqslant j \leqslant m + 1. \tag{4}$$

where the constants

$$0 < L_j < \infty, \quad 1 \leqslant j \leqslant m + 1, \tag{5}$$

are known (this supposition is classical in global optimization (see Hansen and Jaumard, 1995, Horst and Tuy, 1996, Pijavskii, 1972), the problem of estimating the values $L_j, 1 \leqslant j \leqslant m + 1$, is not discussed in this paper). Since the functions $g_j(x), 1 \leqslant j \leqslant m$, are supposed to be multiextremal, the subdomains $Q_j, 2 \leqslant j \leqslant m + 1$, can have a few disjoint subregions each. In the following we shall suppose that all the sets Q_j, $2 \leqslant j \leqslant m+1$, either are empty or consist of a finite number of disjoint intervals of a finite positive length.

In the example shown in Fig. 1(a) the problem (1) has two constraints $g_1(x)$ and $g_2(x)$. The corresponding sets $Q_1 = [a, b]$, Q_2, and Q_3 are shown. It can be seen that the subdomain Q_2 has three disjoint subregions and the constraint $g_2(x)$ is not defined over the subinterval $[c, d]$. The objective function $f(x)$ is defined only over the set Q_3.

The problem (1) may be restated using the *index scheme* proposed originally in Strongin (1984) (see also Strongin and Markin, 1986; Strongin and Sergeyev, 2000). The index scheme does not introduce additional variables and/or parameters by opposition to classical approaches in Bertsekas (1996), Bertsekas (1999), Horst and Pardalos (1995), Horst and Tuy (1996), Nocedal and Wright (1999). It considers constraints one at a time at every point where it has been decided to calculate $g_{m+1}(x)$. Each constraint $g_i(x)$ is evaluated only if all the inequalities

$$g_j(x) \leqslant 0, \qquad 1 \leqslant j < i,$$

have been satisfied.

In its turn the objective function $g_{m+1}(x)$ is computed only for that points where all the constraints have been satisfied.

Let us present the index scheme. Using the designations (2), (3) we can rewrite the problem (1) as the problem of finding a point x_M^* and the corresponding value g_M^* such that

$$g_M^* = g_M(x_M^*) = \min\{g_M(x) : x \in Q_M\}. \tag{6}$$

The values x_M^*, g_M^* coincide with the global solution of the problem (1) if $M = m+1$, i.e., when the original problem is feasible. We associate with every point of the interval $[a, b]$ the *index*

$$\nu = \nu(x), \quad 1 \leqslant \nu \leqslant M,$$

which is defined by the conditions

$$g_j(x) \leqslant 0, \quad 1 \leqslant j \leqslant \nu - 1, \qquad g_\nu(x) > 0, \tag{7}$$

where for $\nu = m+1$ the last inequality is omitted. We shall call *trial* the operation of evaluation of the functions $g_j(x)$, $1 \leqslant j \leqslant \nu(x)$, at a point x. Let us introduce now an auxiliary function $\varphi(x)$ defined over the interval $[a, b]$ as follows

$$\varphi(x) = g_{\nu(x)}(x) - \begin{cases} 0\,, & \text{if } \nu(x) < m+1 \\ g_{m+1}^*\,, & \text{if } \nu(x) = m+1 \end{cases} \tag{8}$$

where g_{m+1}^* is the solution to the problem (1) and to the problem (6) in the case $M = m+1$. Due to (6), (8), the function $\varphi(x)$ has the following properties:

(i) $\varphi(x) > 0$, when $\nu(x) < m+1$;

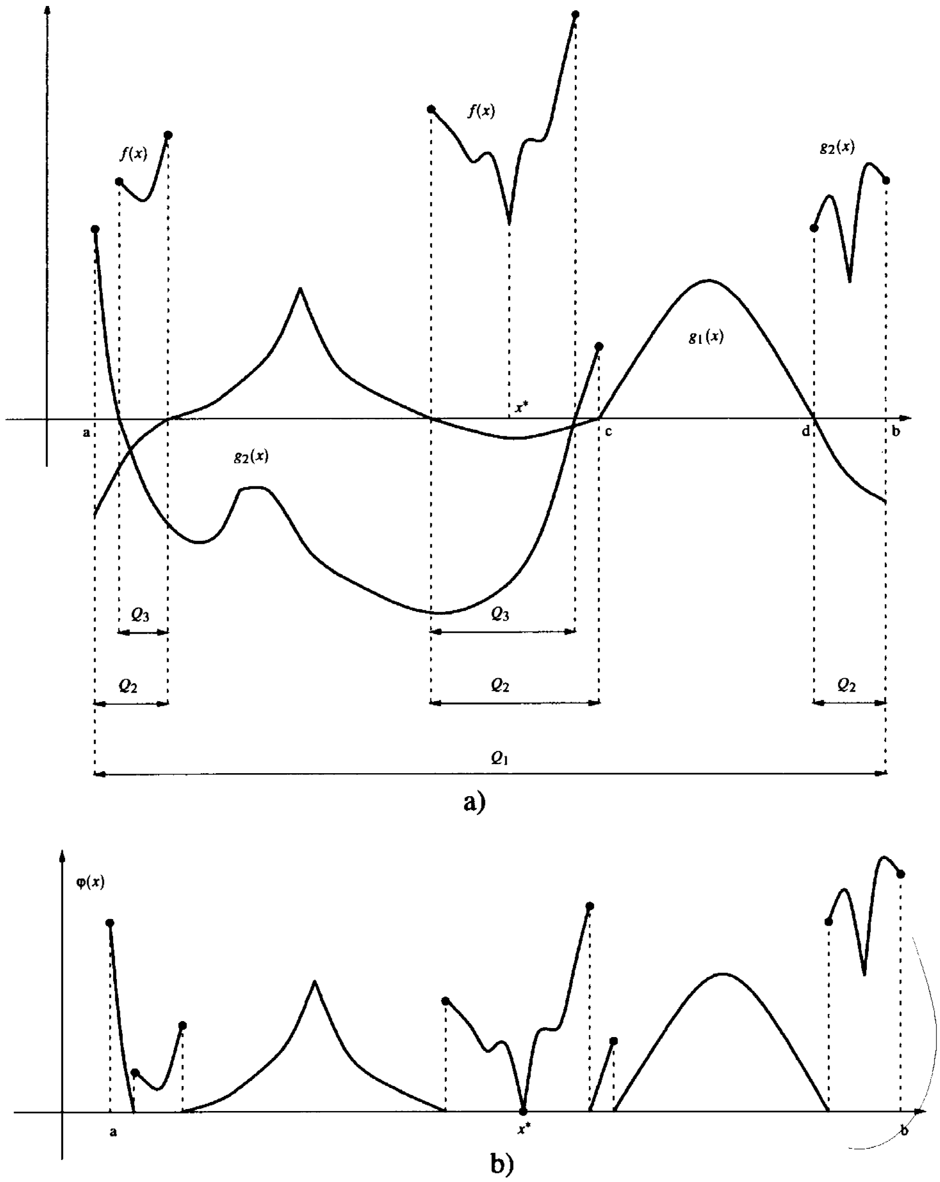

Figure 1. Construction of the function $\varphi(x)$

(ii) $\varphi(x) \geqslant 0$, when $\nu(x) = m + 1$;

(iii) $\varphi(x) = 0$, when $\nu(x) = m + 1$ and $g_{m+1}(x) = g^*_{m+1}$.

In this way the global minimizer of the original constrained problem (1) coincides with the solution x^* of the following unconstrained discontinuous problem

$$\varphi(x^*) = \min\{\varphi(x) : x \in [a, b]\}, \tag{9}$$

in the case $M = m + 1$ and $g_{m+1}(x^*) = g^*_{m+1}$. Obviously, the value g^*_{m+1} used in the construction (8) is not known. Fig. 1(b) shows the function $\varphi(x)$ constructed for the original problem from Fig. 1(a).

Numerical methods belonging to the class of *information algorithms* based on probabilistic ideas have been proposed for solving the problem (9) in Strongin (1984), Sergeyev and Markin (1995), Strongin and Markin (1986), Strongin and Sergeyev (2000).

In this paper a new method called *Index Branch-and-Bound Algorithm (IBBA)* is introduced for solving the discontinuous problem (9). The next section shows that, in spite of the presence of unknown points of discontinuity, it is possible to construct adaptively improved auxiliary functions (called by the authors *index support functions*) for the function $\varphi(x)$ and to obtain lower and upper bounds for the global minimum. The computational scheme of the new method is described in Section 3. Convergence conditions of the algorithm are established in Section 4. Section 5 contains wide computational results showing quite a promising behaviour of the new algorithm. Finally, Section 6 concludes the paper.

2. Discontinuous index support functions

It has been shown in Pijavskii (1972) that lower and upper bounds can be found for the global solution F^* of the problem

$$F^* = \min\{F(x) : x \in [a, b]\}, \tag{10}$$

where

$$| F(x') - F(x'') | \leqslant L_F \, | x' - x'' |, \qquad x', x'' \in [a, b], \tag{11}$$

through sequential updating of a piece-wise linear support function

$$\psi(x) \leqslant F(x), \qquad x \in [a, b], \tag{12}$$

if the Lipschitz constant $0 < L_F < \infty$ is known. The algorithm proposed in (Pijavskii, 1972) improves the support function during every iteration by adding a new point where the objective function $F(x)$ is evaluated. This procedure enables to draw the support function closer to the objective and, therefore, to decrease the gap between the lower and upper bounds. Let us show that by the using index approach it is possible to propose a procedure allowing to obtain lower and upper bounds for the solution g^*_{m+1}. In order to induce the exhaustiveness of the partitioning scheme in the further consideration it is supposed that constants K_j such that

$$L_j < K_j < \infty, \quad 1 \leqslant j \leqslant m + 1, \tag{13}$$

are known. The case $L_j = K_j$ is discarded from the further consideration because in the algorithm of Pijavskii it leads to a possibility of generation of a new point coinciding with one of the points previously generated by the method.

Suppose that k trials have been executed at some points

$$a = x_0 < x_1 < \ldots < x_i < \ldots < x_k = b \tag{14}$$

and the indexes $\nu_i = \nu(x_i), 0 \leqslant i \leqslant k$, have been calculated in accordance with (7). Since the value g^*_{m+1} from (8) is not known, it is not possible to evaluate the function $\varphi(x)$ for the points having the index $m+1$. In order to overcome this difficulty, we introduce the function $\varphi_k(x)$ which is evaluated at the points x_i and gives us the values $z_i = \varphi_k(x_i), 0 \leqslant i \leqslant k$, as follows

$$\varphi_k(x) = g_{\nu(x)}(x) - \begin{cases} 0 & \text{if } \nu(x) < m+1 \\ Z^*_k & \text{if } \nu(x) = m+1 \end{cases} \tag{15}$$

where the value

$$Z^*_k = \min\{g_{m+1}(x_i) : 0 \leqslant i \leqslant k, \nu_i = m+1\}. \tag{16}$$

estimates g^*_{m+1} from (8). It can be seen from (8), (15), and (16) that $\varphi_k(x_i) = \varphi(x_i)$ for all points x_i having indexes $\nu(x_i) < m+1$ and

$$0 \leqslant \varphi_k(x_i) \leqslant \varphi(x_i)$$

if $\nu(x_i) = m+1$. In addition,

$$\varphi_k(x) \leqslant 0, \qquad x \in \{x : g_{m+1}(x) \leqslant Z^*_k\}. \tag{17}$$

During every iteration the trial points $x_i, 0 \leqslant i \leqslant k$, form subintervals

$$[x_{i-1}, x_i] \subset [a, b], \qquad 1 \leqslant i \leqslant k,$$

and every point x_i has its own index $\nu_i = \nu(x_i), 0 \leqslant i \leqslant k$, calculated in accordance with (7). Then, there exist the following three types of subintervals:

(i) intervals $[x_{i-1}, x_i]$ such that $\nu_{i-1} = \nu_i$;
(ii) intervals $[x_{i-1}, x_i]$ such that $\nu_{i-1} < \nu_i$;
(iii) intervals $[x_{i-1}, x_i]$ such that $\nu_{i-1} > \nu_i$.

The bounding procedure presented below constructs over each interval $[x_{i-1}, x_i]$ for the function $\varphi_k(x)$ from (15) a discontinuos *index support function* $\psi_i(x)$ with the following properties

$$\psi_i(x) \leqslant \varphi_k(x), \qquad x \in [x_{i-1}, x_i] \cap Q_{\overline{\nu_i}},$$

where

$$\overline{\nu_i} = \max\{\nu(x_{i-1}), \nu(x_i)\}.$$

Note that the introduced notion is weaker than the usual definition of a support function (cf. (12)). In fact, nothing is required with regard to behaviour of $\psi_i(x)$ over $[x_{i-1}, x_i] \setminus Q_{\overline{\nu_i}}$ and $\psi_i(x)$ can be greater than $\varphi_k(x)$ on this subdomain.

Let us consider one after another the possibilities (i)–(iii). The first case, $\nu_{i-1} = \nu_i$, is the simplest one. Since the indexes of the points x_{i-1}, x_i coincide, the index support function is similar to that one proposed in Pijavskii (1972). In this case, due to (4), (13), and Pijavskii (1972), we can construct for $\varphi_k(x)$ the index support function $\psi_i(x)$, $x \in [x_{i-1}, x_i]$, such that

$$\varphi_k(x) \geqslant \psi_i(x), \qquad x \in [x_{i-1}, x_i] \cap Q_{\nu_i},$$

where the function $\psi_i(x)$ (see (15), (16)) has the form

$$\psi_i(x) = \max\{g_{\nu_i}(x_{i-1}) - K_{\nu_i} \mid x_{i-1} - x \mid, g_{\nu_i}(x_i) - K_{\nu_i} \mid x_i - x \mid\} \tag{18}$$

in the case $\nu_{i-1} = \nu_i < m + 1$ and the form

$$\psi_i(x) = \max\{g_{m+1}(x_{i-1}) - Z_k^* - K_{m+1} \mid x_{i-1} - x \mid,$$

$$g_{m+1}(x_i) - Z_k^* - K_{m+1} \mid x_i - x \mid\} \tag{19}$$

in the case $\nu_{i-1} = \nu_i = m + 1$; the constants K_{ν_i} are from (13). In both cases the global minimum R_i of the function $\psi_i(x)$ over the interval $[x_{i-1}, x_i]$ is

$$R_i = 0.5(z_{i-1} + z_i - K_{\nu_i}(x_i - x_{i-1})), \tag{20}$$

and is reached at the point

$$y_i = 0.5(x_{i-1} + x_i - (z_i - z_{i-1})/K_{\nu_i}). \tag{21}$$

This case is illustrated in Fig. 2 where the points x_i, x_{i+1}, ends of the interval $[x_i, x_{i+1}]$, have the indexes $\nu_i = \nu_{i+1} = j + 1 < m + 1$. In this example

$$g_1(x_i) \leqslant 0, \dots, g_j(x_i) \leqslant 0, \qquad g_{j+1}(x_i) > 0,$$

$$g_1(x_{i+1}) \leqslant 0, \dots, g_j(x_{i+1}) \leqslant 0, \qquad g_{j+1}(x_{i+1}) > 0,$$

$$z_i = \varphi_k(x_i) = g_{j+1}(x_i), \qquad z_{i+1} = \varphi_k(x_{i+1}) = g_{j+1}(x_{i+1}).$$

The values R_{i+1} and y_{i+1} are also shown. The interval $[x_{i-2}, x_{i-1}]$ in the same Figure illustrates the case $\nu_{i-2} = \nu_{i-1} = j$.

The second case is $\nu_{i-1} < \nu_i$. Due to the index scheme, this means that the function $\varphi_k(x)$ has at least one point of discontinuity ω over the interval $[x_{i-1}, x_i]$ (see an example in Fig. 2) and consists of parts having different indexes. To solve the problem (9) we are interested in finding the subregion having the maximal index M from (3). The point x_i has the index $\nu_i > \nu_{i-1}$ and, due to (15), we need an estimate of the minimal value of the function $\varphi_k(x)$ only over the domain $[x_{i-1}, x_i] \cap Q_{\nu_i}$. The right margin of this domain is the point x_i because it is the

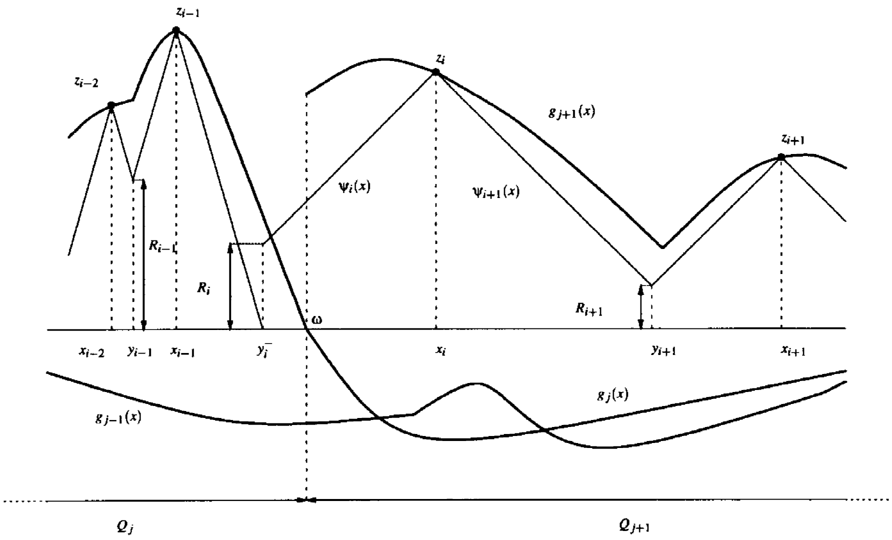

Figure 2. The case $\nu(x_{i-2}) = \nu(x_{i-1}) = j, \quad \nu(x_i) = \nu(x_{i+1}) = j+1$

right end of the interval $[x_{i-1}, x_i]$ and its index is equal to ν_i. It could be possible to take the point x_{i-1} as an estimate of the left margin of the domain $[x_{i-1}, x_i] \cap Q_{\nu_i}$ but a more accurate estimate can be obtained.

It follows from the inequality $\nu_{i-1} < \nu_i$ that

$$z_{i-1} = \varphi_k(x_{i-1}) = g_{\nu_{i-1}}(x_{i-1}) > 0, \qquad g_{\nu_{i-1}}(x_i) \leqslant 0.$$

The function $g_{\nu_{i-1}}(x)$ satisfies the Lipschitz condition, thus

$$g_{\nu_{i-1}}(x) > 0, \qquad x \in [x_{i-1}, y_i^-) \cap Q_{\nu_{i-1}},$$

where the point y_i^- is obtained from (18)

$$y_i^- = x_{i-1} + z_{i-1}/K_{\nu_{i-1}}. \tag{22}$$

An illustration of this situation is given in Fig. 2 where the point $\omega \in [y_i^-, x_i]$ is such that $g_{\nu_{i-1}}(\omega) = 0$ and

$$[x_{i-1}, x_i] \cap Q_{\nu_{i-1}} = [x_{i-1}, \omega], \qquad [x_{i-1}, x_i] \cap Q_{\nu_i} = [\omega, x_i],$$
$$[x_{i-1}, y_i^-] \cap Q_{\nu_{i-1}} = [x_{i-1}, y_i^-].$$

Therefore, the function $g_{\nu_i}(x)$ can be defined at most over the interval $[y_i^-, x_i]$ and the point y_i^- can be used as an estimate of the left margin of the set $[x_{i-1}, x_i] \cap Q_{\nu_i}$ for finding a lower bound for the function $\varphi_k(x)$ over this domain. The corresponding index support function $\psi_i(x)$ in this case has the form

$$\psi_i(x) = z_i - K_{\nu_i} \mid x_i - x \mid \tag{23}$$

and, therefore,

$$\min\{\psi_i(x) : x \in [y_i^-, x_i]\} \leqslant \min\{\psi_i(x) : x \in [y_i^-, x_i] \cap Q_{\nu_i}\}.$$

This minimum is located at the point y_i^- and can be evaluated as

$$R_i = z_i - K_{\nu_i}(x_i - y_i^-) = z_i - K_{\nu_i}(x_i - x_{i-1} - z_{i-1}/K_{\nu_{i-1}}). \tag{24}$$

Let us consider the last case $\nu_{i-1} > \nu_i$ being similar to the previous one. The point x_i has the index $\nu_i < \nu_{i-1}$ and, due to the index scheme, we need an estimate of the minimal value of the function $\varphi_k(x)$ over the domain $[x_{i-1}, x_i] \cap Q_{\nu_{i-1}}$.

Since we have $\nu_{i-1} > \nu_i$, it follows

$$z_i = \varphi_k(x_i) = g_{\nu_i}(x_i) > 0, \qquad g_{\nu_i}(x_{i-1}) \leqslant 0.$$

The function $g_{\nu_i}(x)$ satisfies the Lipschitz condition and, therefore,

$$g_{\nu_i}(x) > 0, \quad x \in (y_i^+, x_i,] \cap Q_{\nu_i},$$

where

$$y_i^+ = x_i - z_i/K_{\nu_i}. \tag{25}$$

Thus, the function $g_{\nu_{i-1}}(x)$ can be defined at most over the interval $[x_{i-1}, y_i^+]$. The corresponding index support function

$$\psi_i(x) = z_{i-1} - K_{\nu_{i-1}} \mid x_{i-1} - x \mid . \tag{26}$$

It is evident that

$$\min\{\psi_i(x) : x \in [x_{i-1}, y_i^+]\} \leqslant \min\{\psi_i(x) : x \in [x_{i-1}, y_i^+] \cap Q_{\nu_{i-1}}\}.$$

It is reached at the point y_i^+ and can be calculated as

$$R_i = z_{i-1} - K_{\nu_{i-1}}(y_i^+ - x_{i-1}) = z_{i-1} - K_{\nu_{i-1}}(x_i - x_{i-1} - z_i/K_{\nu_i}). \tag{27}$$

This case is illustrated in Fig. 3. The points x_{i-1}, x_i have the indexes $\nu_{i-1} = j+2 < m+1,\ \nu_i = j$. This means that

$$g_1(x_{i-1}) \leqslant 0, \dots , g_{j+1}(x_{i-1}) \leqslant 0, \quad g_{j+2}(x_{i-1}) > 0,$$

$$g_1(x_i) \leqslant 0, \dots , g_{j-1}(x_i) \leqslant 0, \quad g_j(x_i) > 0.$$

The values z_{i-1} and z_i are evaluated as follows

$$z_{i-1} = \varphi_k(x_{i-1}) = g_{j+2}(x_{i-1}), \quad z_i = \varphi_k(x_i) = g_j(x_i).$$

Fig. 3 presents a more complex situation in comparison with Fig. 2. In fact,

$$[x_{i-1}, y_i^+] \cap Q_{\nu_{i-1}} = [x_{i-1}, \omega] \setminus \{Q_{j+1} \cap [x_{i-1}, x_i]\}.$$

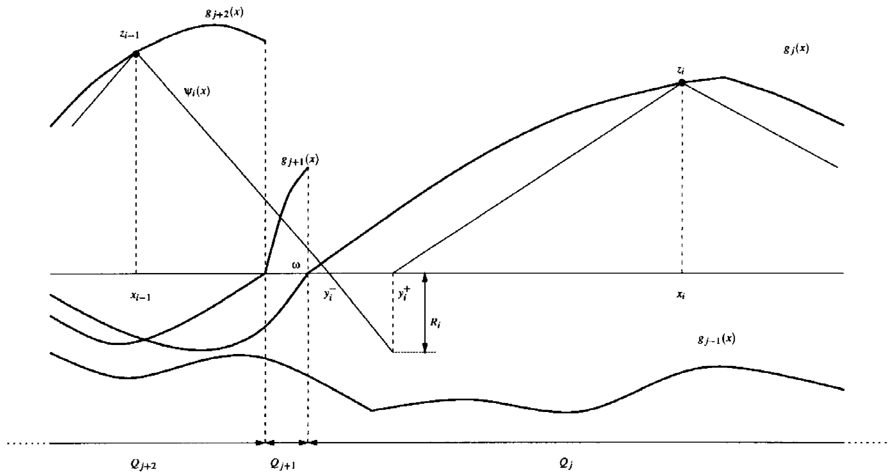

Figure 3. The case $\nu(x_{i-1}) = j + 2, \quad \nu(x_i) = j$

The existence of the subregion $Q_{j+1} \cap [x_{i-1}, x_i] \neq \varnothing$ cannot be discovered by the introduced procedure in the current situation because only the information

$$x_{i-1}, \nu_{i-1}, K_{\nu_{i-1}}, z_{i-1}, \qquad x_i, \nu_i, K_{\nu_i}, z_i$$

regarding the function $\varphi_k(x)$ over $[x_{i-1}, x_i]$ is available. This fact is not relevant because we are looking for subregions with the maximal index M, i.e. subregions where the index is equal to $j + 1$ are not of interest because $M \geqslant j + 2$ since $\nu_{i-1} = j + 2$.

Now we have completed construction of the function $\psi_i(x)$. In all three cases, (i) – (iii), the value R_i being the global minimum of $\psi_i(x)$ over the interval $[x_{i-1}, x_i]$ has been found (hereinafter we call the value R_i *characteristic* of the interval $[x_{i-1}, x_i]$). It is calculated by using one of the formulae (20),(24), or (27) and is reached at the points y_i from (21), y_i^- is from (22), or y_i^+ from (25), correspondingly.

If for an interval $[x_{i-1}, x_i]$ a value $R_i > 0$ has been obtained then, due to the index scheme, it can be concluded that the global solution $x^*_{m+1} \notin [x_{i-1}, x_i]$. For example, in Fig. 2 the intervals

$$[x_{i-2}, x_{i-1}], \quad [x_{i-1}, x_i], \quad [x_i, x_{i+1}]$$

have positive characteristics and, therefore, do not contain the global minimizer.

Let us now consider an interval $[x_{i-1}, x_i]$ of the type (iii) having a negative characteristic (see, for example, the interval $[x_{i-1}, x_i]$ from Fig. 3). The value $R_i < 0$ has been evaluated at the point y_i^+ as the minimum of the function $\psi_i(x)$ from (26). Since $z_{i-1} = \psi_i(x_{i-1}) > 0$ and $R_i = \psi_i(y_i^+) < 0$, a point $\chi \in [x_{i-1}, y_i^+]$ such that $\psi_i(\chi) = 0$ can be found. It follows from (22) that $\chi = y_i^-$. Thus, the subinterval $[y_i^-, y_i^+]$ is the only set over $[x_{i-1}, x_i]$ where the function $\varphi_k(x)$ can be less than zero and where, therefore, the global solution x^*_{m+1} can possibly be located. By analogy, it can be shown that when $R_i < 0$ in the cases (i) and (ii), the interval $[y_i^-, y_i^+]$ is again the only subinterval of $[x_{i-1}, x_i]$ where the global solution can possibly be located.

The *Index Branch-and-Bound Algorithm (IBBA)* proposed in the next section at every $(k+1)$th iteration on the basis of information obtained during the previous k trials constructs the function $\varphi_k(x)$ and the index support functions $\psi_i(x)$, $1 \leqslant i \leqslant k$. Among all the intervals $[x_{i-1}, x_i]$, $1 \leqslant i \leqslant k$, it finds an interval t with the minimal characteristic R_t, and chooses the new trial point x^{k+1} within this interval as follows

$$x^{k+1} = \begin{cases} 0.5(y_t^- + y_t^+), & \nu_{t-1} = \nu_t \\ 0.5(y_t^- + x_t), & \nu_{t-1} < \nu_t \\ 0.5(x_{t-1} + y_t^+), & \nu_{t-1} > \nu_t \end{cases} \tag{28}$$

Note that for intervals having $\nu_{t-1} = \nu_t$ the new trial point x^{k+1} coincides with the Pijavskii point y_i, $i = t$, from (21). Thus, the new algorithm at every iteration updates the function $\varphi_k(x)$ making it closer to $\varphi(x)$ trying to improve the estimate Z_k^* of the global minimum g^*_{m+1}.

3. Description of the algorithm

Let us describe the decision rules of the IBBA. The algorithm starts with two initial trials at the points $x^0 = a$ and $x^1 = b$. Suppose now that: a search accuracy ε has been chosen; k trials have been already done at some points $x^0, \dots, x^k$; their indexes and the value

$$M^k = \max\{\nu(x^i) : 0 \leqslant i \leqslant k\} \tag{29}$$

have been calculated. Here the value M^k estimates the maximal index M from (3).

The choice of the point x^{k+1}, $k \geqslant 1$, at the $(k+1)$-th iteration is determined by the rules presented below.

Step 1. The points $x^0, \dots, x^k$ of the previous k iterations are renumbered by subscripts in order to form the sequence (14). Thus, two numerations are used during the work of the algorithm. The record x^k means that this point has been generated during the k-th iteration of the IBBA. The record x_k indicates the place of the point in the row (14). Of course, the second enumeration is changed during every iteration.

Step 2. Recalculate the estimate Z_k^* from (16) and associate with the points x_i the values $z_i = \varphi_k(x_i)$, $0 \leqslant i \leqslant k$, where the values $\varphi_k(x_i)$ are from (15).

Step 3. For each interval $[x_{i-1}, x_i]$, $1 \leqslant i \leqslant k$, calculate the characteristic of the interval

$$R_i = \begin{cases} 0.5(z_{i-1} + z_i - K_{\nu_i}(x_i - x_{i-1})), & \nu_{i-1} = \nu_i \\ z_i - K_{\nu_i}(x_i - x_{i-1} - z_{i-1}/K_{\nu_{i-1}}), & \nu_{i-1} < \nu_i \\ z_{i-1} - K_{\nu_{i-1}}(x_i - x_{i-1} - z_i/K_{\nu_i}), & \nu_{i-1} > \nu_i \end{cases} \tag{30}$$

Step 4. Find the interval number t such that

$$t = \min\{\arg\min\{R_i : 1 \leqslant i \leqslant k\}\}. \tag{31}$$

Step 5. *(Stopping Rule)* If $R_t > 0$, then Stop (the feasible region is empty). Otherwise, if

$$x_t - x_{t-1} > \varepsilon \tag{32}$$

go to Step 6 (ε is a preset accuracy and t is from (31)). In the opposite case, Stop (the required accuracy has been reached).

Step 6. Execute the $(k+1)$-th trial at the point x^{k+1} from (28), evaluate its index $\nu(x^{k+1})$ and the estimate M^{k+1}, and go to Step 1.

In the following section we will gain more insight the method by establishing and discussing its convergence conditions.

4. Convergence conditions

In this section we demonstrate that the infinite trial sequence $\{x^k\}$ generated by the algorithm IBBA ($\varepsilon = 0$ in the stopping rule) converges to the global solution of the unconstrained problem (9) and, as consequence, to the global solution of the initial constrained problem (1) if it is feasible. In the opposite case the method establishes infeasibility of the problem (1) in a finite number of iterations.

In Lemma 1, we prove the exhaustiveness of the branching scheme. The convergence results of the proposed method can be derived as a particular case of general convergence studies given in Horst and Tuy (1996), Pintér (1996), Sergeyev (1999). We present a detailed and independent proof of these results in Theorems 1 and 2.

LEMMA 1. *Let $\bar{x}$ be a limit point of the sequence $\{x^k\}$ generated by the IBBA with $\varepsilon = 0$ in the stopping rule (32), and let $i = i(k)$ be the number of an interval $[x_{i(k)-1}, x_{i(k)}]$ containing this point during the k-th iteration. Then*

$$\lim_{k\to\infty} x_{i(k)} - x_{i(k)-1} = 0. \tag{33}$$

Proof. During the current k-th iteration an interval $[x_{t-1}, x_t]$ is chosen for subdivision. Due to the decision rules of the IBBA and (22), (25), this means that its characteristic $R_t \leqslant 0$ and the point x^{k+1} from (28) falling into the interval (x_{t-1}, x_t) can be rewritten as follows

$$x^{k+1} = \begin{cases} 0.5(x_t + x_{t-1} + z_{t-1}/K_{\nu_{t-1}} - z_t/K_{\nu_t}), & \nu_{t-1} = \nu_t \\ 0.5(x_t + x_{t-1} + z_{t-1}/K_{\nu_{t-1}}), & \nu_{t-1} < \nu_t \\ 0.5(x_t + x_{t-1} - z_t/K_{\nu_t}), & \nu_{t-1} > \nu_t \end{cases} \tag{34}$$

This point divides the interval $[x_{t-1}, x_t]$ into two subintervals

$$[x_{t-1}, x^{k+1}], \qquad [x^{k+1}, x_t]. \tag{35}$$

Let us show that the following contracting estimate

$$\max\{x_t - x^{k+1}, x^{k+1} - x_{t-1}\} \leqslant$$

$$0.5(1 + \max\{L_{\nu_{t-1}}/K_{\nu_{t-1}}, L_{\nu_t}/K_{\nu_t}\})(x_t - x_{t-1}) \tag{36}$$

holds for the intervals (35), where

$$0.5 \leqslant 0.5(1 + \max\{L_{\nu_{t-1}}/K_{\nu_{t-1}}, L_{\nu_t}/K_{\nu_t}\}) < 1. \tag{37}$$

Let us consider three cases.

(i) In the first case $\nu_{t-1} = \nu_t$, and $z_{t-1} = g_{\nu_t}(x_{t-1})$, $z_t = g_{\nu_t}(x_t)$. It follows from (11), (13) that

$$L_{\nu_{t-1}} = L_{\nu_t} < K_{\nu_{t-1}} = K_{\nu_t},$$

$$| z_t - z_{t-1} | \leqslant L_{\nu_t}(x_t - x_{t-1})$$

and, due to (21), we have

$$\max\{x_t - x^{k+1}, x^{k+1} - x_{t-1}\} \leqslant 0.5(1 + L_{\nu_{t-1}}/K_{\nu_{t-1}})(x_t - x_{t-1}).$$

Thus, (36) and (37) have been established.

(ii) In the second case, $\nu_{t-1} < \nu_t$, and, therefore, due to the index scheme, $z_{t-1} = g_{\nu_{t-1}}(x_{t-1}) > 0$ and $g_{\nu_{t-1}}(x_t) \leqslant 0$. From this estimate and the obvious relation

$$g_{\nu_{t-1}}(x_t) \geqslant z_{t-1} - L_{\nu_{t-1}}(x_t - x_{t-1})$$

we obtain

$$z_{t-1} - L_{\nu_{t-1}}(x_t - x_{t-1}) \leqslant 0. \tag{38}$$

Since $z_{t-1} > 0$, it follows from (38), (28), and (13) that

$$x^{k+1} - x_{t-1} = 0.5(x_t - x_{t-1} + z_{t-1}/K_{\nu_{t-1}}) \leqslant$$

$$0.5(x_t - x_{t-1} + L_{\nu_{t-1}}/K_{\nu_{t-1}}(x_t - x_{t-1})). \tag{39}$$

Let us now estimate the difference $x_t - x^{k+1}$.

$$x_t - x^{k+1} = 0.5(x_t - x_{t-1} - z_{t-1}/K_{\nu_{t-1}}) < 0.5(x_t - x_{t-1}). \tag{40}$$

Obviously, the estimate (36) is the result of (39) and (40).

(iii) The case $\nu_{t-1} > \nu_t$ is considered by a complete analogy to the case (ii) and leads to estimates

$$x^{k+1} - x_{t-1} < 0.5(x_t - x_{t-1}),$$

$$x_t - x^{k+1} \leqslant 0.5(x_t - x_{t-1} + L_{\nu_t}/K_{\nu_t}(x_t - x_{t-1})).$$

To prove (37) it is enough to mention that $L_{\nu_{t-1}}$, $K_{\nu_{t-1}}$, L_{ν_t}, and K_{ν_t} are constants and (13) takes place for them. The result (33) is a straightforward consequence of the decision rules of the IBBA and the estimates (36), (37). □

THEOREM 1. *If the original problem (1) is infeasible then the algorithm stops in a finite number of iterations.*

Proof. If the original problem (1), (4) is infeasible then the maximal index M over the interval $[a, b]$ is less than $m + 1$. In this case (see (7), (8), and (15))

$$\varphi(x) = \varphi_k(x) > 0, \qquad x \in [a, b].$$

On one hand, due to (4), (13), the linear pieces of the index support functions $\psi_i(x), 1 \leqslant i \leqslant k$, from (18), (23), and (26) constructed by the algorithm have a finite slope. On the other hand, Lemma 1 shows that the length of any interval containing any limit point goes to zero.

Thus, it follows from our supposition regarding the sets $Q_j, 1 \leqslant j \leqslant m + 1$, being either empty or consisting of a finite number of disjoint intervals of a finite positive length and the formulae (37), (30), and (31) that there exists a finite iteration number N such that a characteristic $R_{t(N)} > 0$ will be obtained and the algorithm will stop. □

Let us now consider the case when the original problem (1) is feasible. This means that $M = m + 1$ in (6), (8). Let us denote by X^* the set of the global minimizers of the problem (1) and by X' the set of limit points of the sequence $\{x^k\}$ generated by the IBBA with $\varepsilon = 0$ in the stopping rule (32).

THEOREM 2. *If the problem (1) is feasible then $X^* = X'$.*

Proof. Since the problem (1) is feasible, the sets $Q_j, 1 \leqslant j \leqslant m + 1$, are not empty and therefore, due to our hypotheses, they consist of a finite number of disjoint intervals of a finite positive length. This fact together with ε from (32) equal to zero leads to existence of an iteration q during which a point x^q having the index $m + 1$ will be generated. Thus (see (15), (16)), the first value $z_i = 0$ corresponding to the point x^q will be obtained and during all the iterations $k > q$ there will exist at least two intervals having negative characteristics (see (30)).

Let us return to the interval $[x_{i-1}, x_i]$ from Lemma 1 containing a limit point $\bar{x} \in X'$. Since it contains the limit point and the trial points are chosen by the rule (31), its characteristic should be negative too for all iterations $k > q$. Then, by taking into consideration the facts that $z_i \geqslant 0, 1 \leqslant i \leqslant k$, (see (15)) it follows from Lemma 1 and (30), (31) that

$$\lim_{k \to \infty} R_{i(k)} = 0. \tag{41}$$

We can conclude from (41) and $R_{i(k)} < 0, k > q$, that

$$\lim_{k\to\infty} \varphi_k(\bar{x}) = \varphi(\bar{x}) = 0. \tag{42}$$

Let us consider an interval $[x_{j(k)-1}, x_{j(k)}]$ containing a global minimizer $x^* \in X^*$ during an iteration $k > q$. At first, we show that there will exist an iteration number $c \geqslant q$ such that $\nu_{j(c)-1} = m+1$ or $\nu_{j(c)} = m+1$. If trials will fall within the interval $[x_{j(k)-1}, x_{j(k)}]$, due to the decision rules of the IBBA, such a trial will be generated. Suppose that trials will not fall into this interval and

$$\Gamma = \max\{\nu(x_{j(k)-1}), \nu(x_{j(k)})\} < m + 1.$$

The point x^* is feasible, this means that

$$x^* \in [\alpha, \beta] = [x_{j(k)-1}, x_{j(k)}] \cap Q_{m+1},$$

where the interval $[\alpha, \beta]$ has a finite positive length and

$$g_l(x) \leqslant 0, \quad 1 \leqslant l \leqslant m, \quad x \in [\alpha, \beta].$$

We obtain from these inequalities that, due to (30) and (13), the characteristic

$$R_{j(k)} \leqslant \min\{g_\Gamma(x) : x \in [\alpha, \beta]\} < 0. \tag{43}$$

Since trials do not fall at the interval $[x_{j(k)-1}, x_{j(k)}]$, it follows from (30) that $R_{j(k)}$ is not changed from iteration to iteration. On the other hand, the characteristic $R_{i(k)} \to 0$ when $k \to \infty$. This means that at an iteration number $k' > q$ the characteristic of the interval $[x_{i-1}, x_i]$, $i = i(k')$, will not be minimal. Thus, a trial will fall into the interval $[x_{j-1}, x_j]$. The obtained contradiction proves generation of a point

$$x^c \in [\alpha, \beta], \quad c \geqslant k', \quad \nu(x^c) = m + 1.$$

We can now estimate the characteristic $R_{j(k)}$ of the interval $[x_{j(k)-1}, x_{j(k)}]$ containing the global minimizer $x^* \in X^*$ during an iteration $k > c$. We have shown that at least one of the points $x_{j(k)-1}, x_{j(k)}$ will have the index $m + 1$. Again, three cases can be considered.

In the case $\nu_{j-1} = \nu_j = m + 1$ it follows from (4), (15) that

$$z_{j-1} - \varphi_k(x^*) \leqslant L_{m+1}(x^* - x_{j-1}).$$

From (17) we have $\varphi_k(x^*) \leqslant 0$ and, therefore,

$$z_{j-1} \leqslant L_{m+1}(x^* - x_{j-1}),$$

Analogously, for the value z_j it follows

$$z_j \leqslant L_{m+1}(x_j - x^*).$$

From these two estimates we obtain

$$z_j + z_{j-1} \leqslant L_{m+1}(x_j - x_{j-1}).$$

By using (13), (20), and the last inequality we deduce

$$R_{j(k)} \leqslant (L_{m+1} - K_{m+1})(x_{j(k)} - x_{j(k)-1}) < 0. \tag{44}$$

Analogously, it can be seen from (22) and (24) that in the case $\nu_{j-1} < \nu_j$ the estimate

$$R_{j(k)} \leqslant (L_{m+1} - K_{m+1})(x_{j(k)} - y^-_{j(k)}) < 0 \tag{45}$$

takes place because

$$z_j \leqslant L_{m+1}(x_j - x^*) \leqslant L_{m+1}(x_{j(k)} - y^-_{j(k)}).$$

For the case $\nu_{j-1} > \nu_j$ (see (25) and (27)) we have

$$R_{j(k)} \leqslant (L_{m+1} - K_{m+1})(y^+_{j(k)} - x_{j(k)-1}) < 0. \tag{46}$$

It follows from (43) – (46) that the characteristic of the interval $[x_{j-1}, x_j]$ containing the global minimizer x^* will be always negative. Assume now, that x^* is not a limit point of the sequence $\{x^k\}$, then there exists a number P such that for all $k \geqslant P$ the interval $[x_{j-1}, x_j]$, $j = j(k)$, is not changed, i.e., new points will not fall into this interval and, as a consequence, its characteristic $R_{j(k)}$ will not change too.

Consider again the interval $[x_{i-1}, x_i]$ from Lemma 1 containing a limit point $\bar{x} \in X'$. It follows from (41) and the fact that $R_{j(k)}$ is a negative constant that there exists an iteration number N such that

$$R_{j(N)} < R_{i(N)}.$$

Due to decision rules of the IBBA, this means that a trial will fall into the interval $[x_{j-1}, x_j]$. But this fact contradicts our assumption that x^* is not a limit point.

Suppose now that there exists a limit point $\bar{x} \in X'$ such that $\bar{x} \notin X^*$. This means that $\varphi(\bar{x}) > \varphi(x^*)$, $x^* \in X^*$. Impossibility of this fact comes from (41), (42), and the fact of $x^* \in X'$. □

We can conclude that if the algorithm has stopped and has not established that $Q_{m+1} = \emptyset$ then the following situations are possible:

(i) If $M^k < m + 1$, then this means that the accuracy ε was not sufficient for establishing the feasibility of the problem;

(ii) If $M^k = m + 1$ and all the intervals $[x_{p-1}, x_p]$ such that

$$\max\{\nu_{p-1}, \nu_p\} < m + 1 \tag{47}$$

have positive characteristics then, we can conclude that the global minimum z^* of the original problem (1) can be bounded as follows

$$z^* \in [R_{t(k)} + Z_k^*, Z_k^*],$$

where the value Z_k^* is from (16) and $R_{t(k)}$ is the characteristic corresponding to the interval number $t = t(k)$ from (31).

(iii) If $M^k = m + 1$ and there exists an interval $[x_{p-1}, x_p]$ such that $R_p \leqslant 0$ and (47) takes place then, the value Z_k^* can be taken as an upper bound of the global minimum z^* of the original problem (1). A rouge lower bound can be calculated easily by taking the trial points x_i such that $\nu(x_i) = m + 1$ and constructing for $f(x)$ the support function of the type (Pijavskii, 1972) using only these points. The global minimum of this support function over the search region $[a, b]$ will be a lower bound for z^*. A more precise lower bound can be obtained by minimizing this support function over the set

$$\bigcup [x_{i-1}, x_i], \qquad R_i < 0, \quad 1 \leqslant i \leqslant k.$$

We do not discuss here the peculiarities of the implementation of the IBBA. Let us make only two remarks. First, it is not necessary to re-calculate all the characteristics during Step 3 but it is sufficient to do this operation only for two new intervals generated during the previous iteration. Second, as it follows from the proofs of Theorems 1 and 2, it is possible to exclude from consideration all the intervals having positive characteristics.

5. Numerical comparison

The IBBA algorithm has been numerically compared to the method (indicated hereinafter as *PEN*) proposed by Pijavskii (see Pijavskii, 1972, Hansen and Jaumard, 1995) combined with a penalty function. The PEN has been chosen for comparison because it uses the same information about the problem as the IBBA — the Lipschitz constants for the objective function and constraints.

Ten differentiable and ten non-differentiable test problems introduced in (Famularo et al. 2001) have been used. In addition, the IBBA has been applied to one differentiable and one non-differentiable infeasible test problem from (Famularo et al. 2001). Since the order of constraints can influence speed of the IBBA significantly, it has been chosen the same as in (Famularo et al. 2001), without determining the best order for the IBBA. The same accuracy $\varepsilon = 10^{-4}(b - a)$ (where b and a are from (1)) has been used in all the experiments for both methods.

In Table I (Differentiable problems) and Table II (Non-Differentiable problems) the results obtained by the IBBA have been summarized and the columns in the Tables have the following meaning:

– the columns XIBBA and FIBBA represent the estimate to the global solution $(x^*, f(x^*))$ found by the IBBA for each problem;

Table I. Results of the experiments executed by the IBBA with the differentiable problems.

Problem	XIBBA	FIBBA	N_{g_1}	N_{g_2}	N_{g_3}	N_f	Iterations	Eval.
1	1.05726259	−7.61226549	10	–	–	13	23	36
2	1.01559921	5.46160556	206	–	–	21	227	248
3	−5.99182849	−2.94266082	40	–	–	22	62	84
4	2.45965829	2.84080900	622	156	–	175	953	1459
5	9.28501542	−1.27484676	8	14	–	122	144	402
6	2.32396546	−1.68515824	14	80	–	18	112	228
7	−0.77473979	−0.33007410	35	18	–	241	294	794
8	−1.12721979	−6.60059664	107	43	5	82	237	536
9	4.00046339	1.92220990	7	36	6	51	100	301
10	4.22474504	1.47400000	37	15	195	1173	1420	5344
Average	–	–	–	–	–	–	357.2	943.2

Table II. Results of the experiments executed by the IBBA with the non-differentiable problems.

Problem	XIBBA	FIBBA	N_{g_1}	N_{g_2}	N_{g_3}	N_f	Iterations	Eval.
1	1.25830963	4.17420017	23	–	–	28	51	79
2	1.95967593	−0.07915191	18	–	–	16	34	50
3	9.40068508	−4.40068508	171	–	–	19	190	209
4	0.33286804	3.34619770	136	15	–	84	235	418
5	0.86995142	0.74167893	168	91	–	24	283	422
6	3.76977185	0.16666667	16	16	–	597	629	1839
7	5.20120767	0.90312158	63	18	–	39	120	216
8	8.02835033	4.05006890	29	11	3	21	64	144
9	0.95032461	2.64804102	8	86	57	183	334	1083
10	0.79996352	1.00023345	42	3	17	13	75	151
Average	–	–	–	–	–	–	201.5	461.1

- the columns N_{g_1}, N_{g_2}, N_{g_3} represent the number of trials where the constraint g_i, $1 \leqslant i \leqslant 3$, was the last evaluated constraint;
- the column N_f shows how many times the objective function $f(x)$ has been evaluated;
- the column 'Eval.' is the total number of evaluations of the objective function and the constraints. This quantity is equal to:
  - $N_{g_1} + 2 \times N_f$, for problems with one constraint;
  - $N_{g_1} + 2 \times N_{g_2} + 3 \times N_f$, for problems with two constraints;
  - $N_{g_1} + 2 \times N_{g_2} + 3 \times N_{g_3} + 4 \times N_f$, for problems with three constraints.

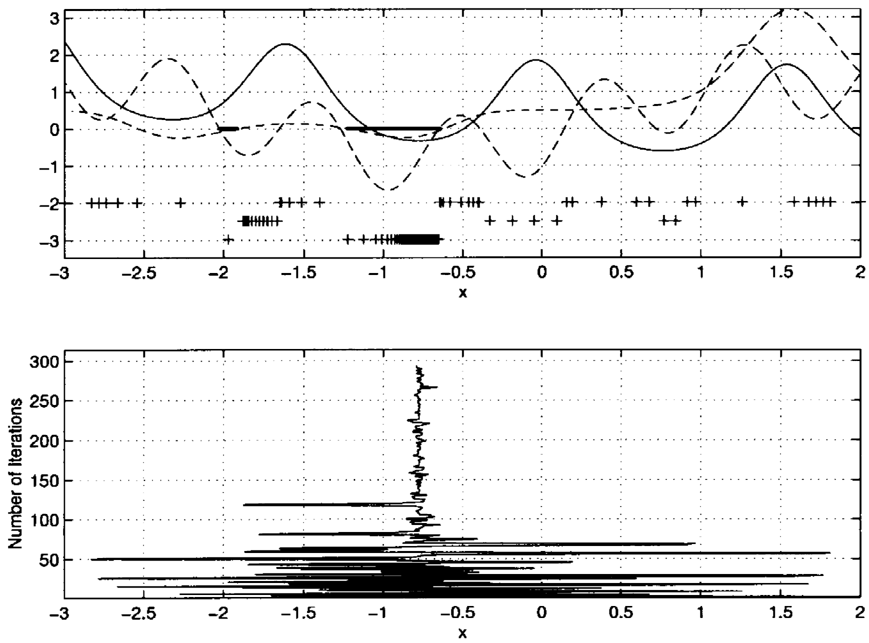

Figure 4. Optimization of the differentiable problem 7 by the IBBA.

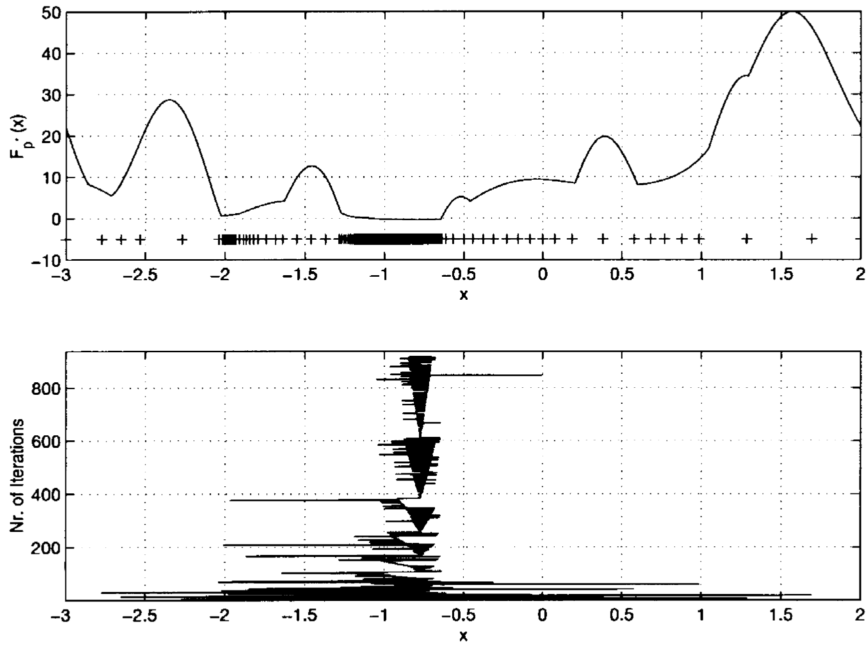

Figure 5. Optimization of the differentiable problem 7 by the PEN.

Table III. Differentiable functions. Numerical results obtained by the PEN.

Problem	XPEN	FXPEN	P^*	Iterations	Eval.
1	1.05718004	−7.61185807	15	83	166
2	1.01609254	5.46142698	90	954	1906
3	−5.99184997	−2.94292577	15	119	238
4	2.45953057	2.84080890	490	1762	5286
5	9.28468704	−1.27484673	15	765	2295
6	2.32334492	−1.68307049	15	477	1431
7	−0.77476915	−0.33007412	15	917	2751
8	−1.12719146	−6.60059658	15	821	3284
9	4.00042801	1.92220821	15	262	1048
10	4.22482084	1.47400000	15	2019	8076
Average	–	–	–	817.9	2648.1

Table IV. Non-Differentiable problems. Numerical results obtained by the PEN.

Problem	XPEN	FXPEN	P^*	Iterations	Eval.
1	1.25810384	4.17441502	15	247	494
2	1.95953624	−0.07902265	15	241	482
3	9.40072023	−4.40072023	15	797	1594
4	0.33278550	3.34620350	15	272	819
5	0.86995489	0.74168456	20	671	2013
6	3.76944805	0.16666667	15	909	2727
7	5.20113260	0.90351752	15	199	597
8	8.02859874	4.05157770	15	365	1460
9	0.95019236	2.64804101	15	1183	4732
10	0.79988668	1.00072517	15	135	540
Average	–	–	–	501.9	1545.8

In Table III (Differentiable problems) and Table IV (Non-Differentiable problems) the results obtained by the PEN are collected. The constrained problems were reduced to the unconstrained ones as follows

$$f_{P^*}(x) = f(x) + P^* \max\left\{g_1(x), g_2(x), \ldots, g_{N_v}(x), 0\right\}. \tag{48}$$

The coefficient P^* has been computed by the rules:

(1) the coefficient P^* has been chosen equal to 15 for all the problems and it has been checked if the found solution (XPEN,FXPEN) for each problem belongs or not to the feasible subregions;

(2) if it does not belong to the feasible subregions, the coefficient P^* has been iteratively increased by 10 starting from 20 until a feasible solution has been found. Particularly, this means that a feasible solution has not been found in Table III for the problem 2 when P^* was equal to 80, for the problem 4 when P^* was equal to 480, and in Table IV for the problem 5 when P^* was equal to 15.

It must be noticed that in Tables III and IV the meaning of the column 'Eval.' is different in comparison with Tables I and II. In Tables III and IV this column shows the total number of evaluations of the objective function $f(x)$ and *all* the constraints. Thus, it is equal to

$$(N_v + 1) \times N_{iter},$$

where N_v is the number of constraints and N_{iter} is the number of iterations for each problem.

In Figures 4 and 5 and Tables V and VI we show the dynamic diagrams of the search executed by the IBBA and the PEN for the differentiable problem 7 from (Famularo et al. 2001):

$$\min_{x\in[-3,2]} f(x) = \exp\left(-\cos\left(4x - 3\right)\right) + \frac{1}{250}\left(4x - 3\right)^2 - 1$$

subject to

$$\begin{aligned} g_1(x) &= \sin^3(x)\exp(-\sin(3x)) + \frac{1}{2} \leqslant 0 \\ g_2(x) &= \cos\left(\frac{7}{5}(x+3)\right) - \sin\left(7(x+3)\right) + \frac{3}{10} \leqslant 0 \end{aligned}$$

The problem has two disjoint feasible subregions shown by two continuous bold lines and the global optimum x^* is located at the point $x^* = -0.774575$.

The first line (from up to down) of '+' located under the graph of the problem 7 in the upper subplot of Figure 4 represents the points where the first constraint has not been satisfied (number of iterations equal to 35). Thus, due to the decision rules of the IBBA, the second constraint has not been evaluated at these points. The second line of '+' represents the points where the first constraint has been satisfied but the second constraint has been not (number of iterations equal to 18). In these points both constraints have been evaluated but the objective function has been not. The last line represents the points where both the constraints have been satisfied (number of evaluations equal to 241). The total number of evaluations is equal to $35 + 18 \times 2 + 241 \times 3 = 794$. These evaluations have been executed during $35 + 18 + 241 = 294$ iterations.

The line of '+' located under the graph in the upper subplot of Figure 5 represents the points where the function (48) has been evaluated. The number of iterations made by the PEN is equal to 917 and the number of evaluations is equal to $917 \times 3 = 2757$.

Table V. Differentiable problems: comparison between the IBBA and the PEN.

Problem	Iterations			Evaluations		
	PEN	IBBA	Speedup	PEN	IBBA	Speedup
1	83	23	3.61	166	36	4.61
2	954	227	4.20	1906	248	7.69
3	119	62	1.92	238	84	2.83
4	1762	953	1.85	5286	1459	3.62
5	765	144	5.31	2295	402	5.71
6	477	112	4.26	1431	228	6.28
7	917	294	3.12	2751	794	3.46
8	821	237	3.46	3284	536	6.13
9	262	100	2.62	1048	301	3.48
10	2019	1420	1.42	8076	5344	1.51
Average	817.9	357.2	2.29	2648.1	943.2	2.81

Finally, the infeasibility of the differentiable problem from (Famularo et al. 2001) has been determined by the IBBA in 38 iterations consisting of nine evaluations of the first constraint and 29 evaluations of the first and second constraints (i.e., 67 evaluations in total). The infeasibility of the non-differentiable problem from (Famularo et al. 2001) has been determined by the IBBA in 98 iterations consisting of 93 evaluations of the first constraint and five evaluations of the first and second constraints (i.e., 103 evaluations in total). Naturally, the objective functions were not evaluated at all in both cases. Note that experiments for infeasible problems have not been executed with the PEN because the penalty approach does not allow to the user to determine infeasibility of problems.

6. Concluding remarks

Lipschitz univariate constrained global optimization problems where both the objective function and constraints can be multiextremal have been considered in this paper. The constrained problem has been reduced to a discontinuous unconstrained problem by the index scheme. A Branch-and-Bound method for solving the reduced problem has been proposed. Convergence conditions of the new method have been established.

The new algorithm works without usage of derivatives. It either determines the infeasibility of the original problems or finds upper and lower bounds of the global solution. Note that it is able to work with problems where the objective function and/or constraints are not defined over the whole search region. It does not evaluate

Table VI. Non-differentiable problems: comparison between the the IBBA and the PEN.

	Iterations			Evaluations		
Problem	PEN	IBBA	Speedup	PEN	IBBA	Speedup
1	247	51	4.84	494	79	6.25
2	241	34	7.09	482	50	9.64
3	797	190	4.19	1594	209	7.63
4	272	235	1.16	819	418	1.96
5	671	283	2.37	2013	422	4.77
6	909	629	1.45	2727	1839	1.48
7	199	120	1.66	597	216	2.76
8	365	64	5.70	1460	144	10.14
9	1183	334	3.54	4732	1083	4.37
10	135	75	1.80	540	151	3.58
Average	501.9	201.5	2.49	1545.8	461.1	3.35

all the constraints during every iteration. The introduction of additional variables and/or parameters is not required.

Extensive numerical results show quite a satisfactory performance of the new technique. The behaviour of the Index Branch-and-Bound method was compared to the method of Pijavskii combined with a penalty approach. This algorithm has been chosen for comparison because it used the same information about the problem as the IBBA — the Lipschitz constants for the objective function and constraints.

A priori the penalty approach combined with the method of Pijavskii seemed to be more attractive because it dealt only with one function. In the facts however, the evaluation of this function requires the evaluation of $m + 1$ initial functions. The second disadvantage of the penalty approach is that it requires an accurate tuning of the penalty coefficient in contrast to the IBBA which works without any additional parameter. Finally, when the penalty approach is used and a constraint $g(x)$ is defined only over a subregion $[c, d]$ of the search region $[a, b]$, the problem of extending $g(x)$ to the whole region $[a, b]$ arises. In contrast, the IBBA does not have this difficulty because every constraint (and the objective function) is evaluated only within its region of definition.

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